Design of Microwave Dielectric Resonators

J. C. SETHARES, MEMBER, IEEE, AND S. J. NAUMANN

Abstract—The resonant frequencies for the fundamental modes in circular cylindrical and rectangular parallelopiped high dielectric resonators have been calculated by computer for a range of values of physical dimensions and relative dielectric constant. The frequency range extends from zero to 50 kMc/s, the relative dielectric constant from 50 to 1800, and physical dimensions from zero to 500 mils. Results are presented in graphical form with frequency plotted vs. resonator length for parametric values of relative dielectric constant and cross-sectional dimensions. A brief review of earlier work with high dielectric resonators is included. Expressions for the resonant frequency and fundamental mode field configurations are given.

Introduction

THE EXISTENCE of low-loss high dielectric ($\epsilon > 100$) resonators has been known for some time, [1] but recently they have received a renewed interest in connection with microwave studies [2]–[4]. Richtmyer [1], showed that a high dielectric material in free space will exhibit radiation damping; but if $\epsilon \gg 1$, the relative damping is small enough to allow the dielectric to resonate. He investigated, theoretically, the resonant properties of toroidal, spherical, and ring shaped dielectric materials. A. Okaya [2] reported X-band unloaded Q's of 9000 at room temperature for rutile resonators. Loaded Q was found to increase with decreasing temperature and at 4.2°K a loaded Q of 10 000 was obtained. When the temperature was lowered from 300°K to 4.2°K the relative dielectric constant increased by 30 percent. Bell and Rupprecht [3] reported experimental results on the loss tangent of strontium titanate ($SrTiO_3$) vs. temperature (−200°C to 250°C) at about 20 kMc/s. The measured loss tangent was on the order of 10^{-3} over the entire temperature range. The dielectric constant of strontium titanate is larger than that of rutile by roughly a factor of three. At room temperature $\epsilon \approx 300$ and at −180°C, $\epsilon \approx 1500$ for strontium titanate.

Okaya and Barash [4] made a detailed analysis of the resonant modes in anisotropic high dielectrics. They identified field configurations of various modes of oscillation and their respective resonant frequencies. They concluded that, basically, two types of fundamental modes exist; one resembles an electric dipole (E mode) and the other a magnetic dipole (H mode) field. The higher modes resemble multipoles. Some experimental results were given of the relative dielectric constant of rutile and strontium titanate as a function of temperature from 100°K to 300°K. For both materials, Q and ϵ were found to increase with decreasing temperature.

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The authors are with the Air Force Cambridge Research Laboratories, Bedford, Mass.

A recent practical investigation of high dielectric resonators has been conducted by Yee [5]. He calculated the exact resonant modes of a spherical resonator, and the approximate ones for circular cylindrical and rectangular parallelopiped structures, assuming dielectric isotropy. The calculated approximate frequencies agree reasonably well with experiment. Also studied were the effects of metallic waveguides into which a resonator is inserted, field strength of lower order modes, frequency tuning techniques, methods of coupling to the resonator, and methods of suppressing unwanted modes. It should be pointed out that Okaya and Yee define E and H modes in the same way, but TE and TM definitions are reversed. Okaya defines E modes as TE and H modes as TM, whereas Yee defines E and H modes as TM and TE modes, respectively. We have adopted Yee's convention.

High dielectric resonators seem to be ideally suited for microwave investigations involving ferromagnetic materials because of their miniature size, low loss, insensitivity to magnetic dc biasing field, and ability to concentrate large RF magnetic fields in small volumes. A rutile dielectric resonator was used, for example, by Shaw [6] to generate 2.8 Gc/s phonons at room temperature in a 4000 Å nickel film deposited on a phonon supporting substrate such as sapphire or rutile. In Shaw's work, rutile was used both as a microwave resonator and phonon transmission line. (The low-frequency acoustic properties of rutile were investigated earlier by Vick and Hollander [7] at 10 Mc/s.)

One disadvantage of rutile and strontium titanate dielectric resonators is their sensitivity to heat. The dielectric constant is strongly temperature dependent which results in a continuous resonant frequency shift under CW operation, even at very low (milliwatt) power levels. Stiglitz and Sethares [8] have reported a method of heat sinking with Boron Nitride which stabilizes the resonator very quickly with 100 mW power dissipation in the resonator.

Theoretical Considerations

According to Yee [8], the frequencies of dominant TE modes for rectangular ($TE_{11\delta}$) and circular cylindrical ($TE_{10\delta}$) resonators, respectively, are given by

$$f^2 = \frac{c^2}{4\epsilon}\left[\left(\frac{1}{A^{*2}} + \frac{1}{B^{*2}}\right) + \frac{\delta^2}{L^2}\right] \tag{1}$$

and

$$f^2 = \frac{c^2}{4\epsilon}\left[\frac{\beta^2}{\pi^2} + \frac{\delta^2}{L^2}\right] \tag{2}$$

where $c=3\times10^8$ m/s, ϵ is the relative dielectric constant, A^* and B^* are cross-sectional rectangular dimensions, L the length of either resonator, $\delta(0<\delta<1)$ denotes a fraction of half wavelength in the L direction, β satisfies $J_0(\beta A)=0$, A is the cylinder radius, and J_0 the zero order Bessel function. The two modes are shown in Fig. 1. The expressions for ***E*** and ***H*** fields are as follows.

For the rectangular $TE_{11\delta}$ mode and $e^{+j\omega t}$ time dependence

$$
\begin{aligned}
H_z &= H_0 \cos\left(\frac{\pi x}{A^*}\right)\cos\left(\frac{\pi y}{B^*}\right)\cos\left(\frac{\pi\delta z}{L}\right)\\
H_y &= \frac{+\delta H_0 G}{B^* L}\cos\left(\frac{\pi x}{A^*}\right)\sin\left(\frac{\pi y}{B^*}\right)\sin\left(\frac{\pi\delta z}{L}\right)\\
H_x &= +\frac{\delta G H_0}{A^* L}\sin\left(\frac{\pi x}{A^*}\right)\cos\left(\frac{\pi y}{B^*}\right)\sin\left(\frac{\pi\delta z}{L}\right)\\
E_x &= \frac{+j\omega\mu_0 H_0 G}{B^*\pi}\cos\left(\frac{\pi x}{A^*}\right)\sin\left(\frac{\pi y}{B^*}\right)\cos\left(\frac{\pi\delta z}{L}\right)\\
E_y &= \frac{-j\omega\mu_0 G H_0}{A^*\pi}\sin\left(\frac{\pi x}{A^*}\right)\cos\left(\frac{\pi y}{B^*}\right)\cos\left(\frac{\pi\delta z}{L}\right) \qquad (3)\\
E_z &= 0
\end{aligned}
$$

where the x, y, z coordinate system origin is located at the geometrical center of the resonator, and

$$\frac{1}{G}=\frac{1}{(A^*)^2}+\frac{1}{(B^*)^2}\,.$$

For the circular cylindrical $TE_{10\delta}$ mode and $e^{+j\omega t}$ dependence

$$
\begin{aligned}
H_z &= H_0 J_0(\beta\rho)\cos\left(\frac{\delta\pi z}{L}\right)\\
H_\rho &= \frac{-\delta\pi H_0}{\beta L}J_0'(\beta\rho)\sin\left(\frac{\delta\pi z}{L}\right)\\
E_\phi &= \frac{+j\omega\mu_0 H_0}{\beta}J_0'(\beta\rho)\cos\left(\frac{\delta\pi z}{L}\right)\\
H_\phi &= E_\rho = E_z = 0 \qquad (4)
\end{aligned}
$$

where the prime denotes differentiation with respect to $(\beta\rho)$.

The preceding results are derived using the assumptions that $H_z=0$ at all surfaces parallel to the z axis (perfect open-circuit boundary conditions); the tangential E and H fields are continuous across surfaces perpendicular to the z axes; the resonators are dielectrically isotropic; and fields outside the resonator decay exponentially from their value at the boundary to zero at infinity. The quantity δ is constrained by boundary conditions on E and H. Using Yee's results, (12), (13), and (14) in this article, δ/L satisfies the following equation

$$\frac{\delta^2}{L^2}=\frac{4f^2}{c^2}(\epsilon-1)\cos^2\left(\frac{\pi\delta}{2}\right) \qquad (5)$$

where

$$\delta\equiv\frac{L\zeta}{\pi}$$

and where $0<\delta<1$ for both rectangular and cylindrical resonators. In addition, the $TE_{10\delta}$ cylindrical mode is dominant (lowest frequency) only if $A/L>0.48$, or if

$$L\leqq\text{cylinder diameter.}$$

The resonant frequencies for $TE_{11\delta}$ and $TE_{10\delta}$ modes are given by (1) and (2), respectively, subject to constraint equation (5). Alternatively, the resonant frequencies for either resonator are given by (2) along with constraint equation (5), if, for the cylindrical $TE_{10\delta}$ mode, β satisfies the first zero of $J_0(\beta A)$, and for the rectangular $TE_{11\delta}$ mode,

$$\frac{\beta^2}{\pi^2}=\frac{1}{G}=\frac{1}{(A^*)^2}+\frac{1}{(B^*)^2}\,.$$

If (5) is substituted into (2), the resonant frequency for either resonator is found to be

$$f=\left(\frac{c\beta}{2\pi}\right)\frac{1}{\sqrt{\epsilon\sin^2\left(\frac{\pi\delta}{2}\right)+\cos^2\left(\frac{\pi\delta}{2}\right)}}\equiv\frac{c\beta}{2\pi}F(\epsilon,\delta) \qquad (7)$$

or

$$f\begin{pmatrix}\text{Rectangular}\\ TE_{11\delta}\end{pmatrix}=\frac{c}{2}\sqrt{\frac{1}{(A^*)^2}+\frac{1}{(B^*)^2}}\,F(\epsilon,\delta) \qquad (8)$$

$$f\begin{pmatrix}\text{Cylindrical}\\ TE_{10\delta}\end{pmatrix}=\left(\frac{(2.405)c}{2\pi A}\right)F(\epsilon,\delta) \qquad (9)$$

where $F(\epsilon,\delta)$ is defined by (7).

Equations (8) and (9) determine the frequencies in terms of cross-sectional dimensions, dielectric constant, and quantity δ; L is not explicitly contained in the equations. Because of this, and since δ prescribes the fractional half wavelength of field variation in the L direction, the field strength at top and bottom resonator surfaces is somewhat adjustable through the cross-sectional dimensions and dielectric constant.

The maximum and minimum allowable frequencies for both modes follow from the asymptotic expressions of (7) as δ approaches zero and one.

$$f_{\delta\to0}=\frac{c\beta}{2\pi}$$

$$f_{\delta\to1}=\frac{c\beta}{2\pi\sqrt{\epsilon}}\,.$$

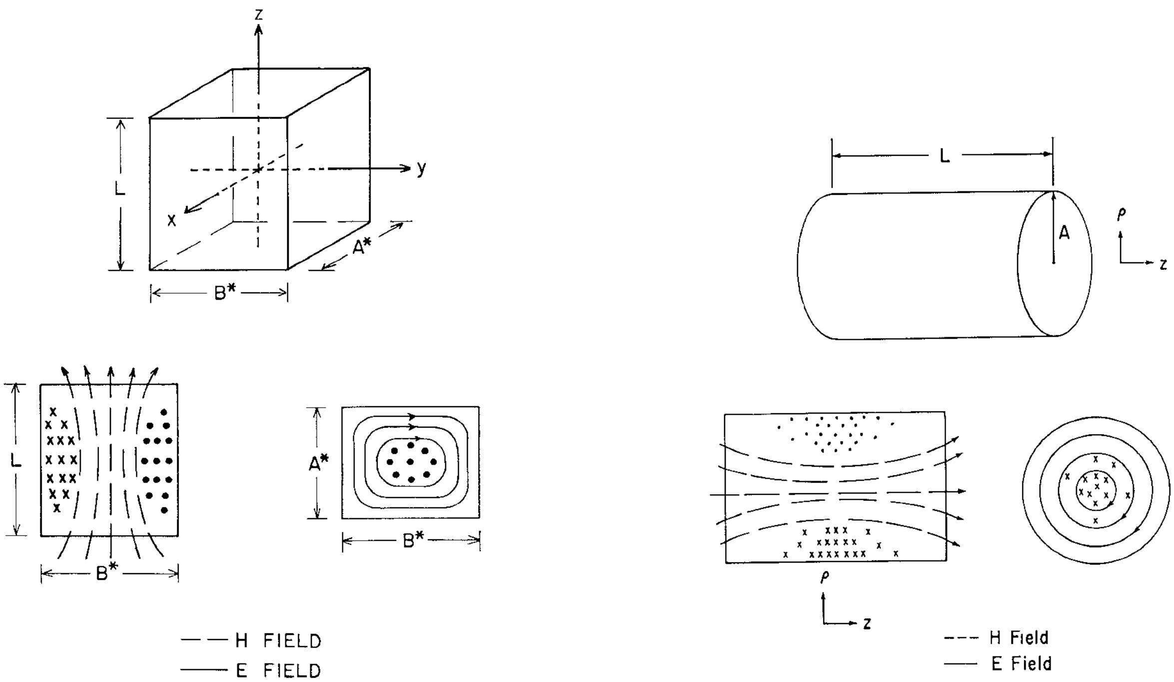
Therefore

$$\frac{c\beta}{2\pi\sqrt{\epsilon}} < f < \frac{c\beta}{2\pi} \cdot \tag{10}$$

Graphs of maximum and minimum allowable frequencies are shown in Fig. 2. Using Yee's equations (12), (13), and (14), the following expression is obtained [10] which relates f, L, ϵ, and β

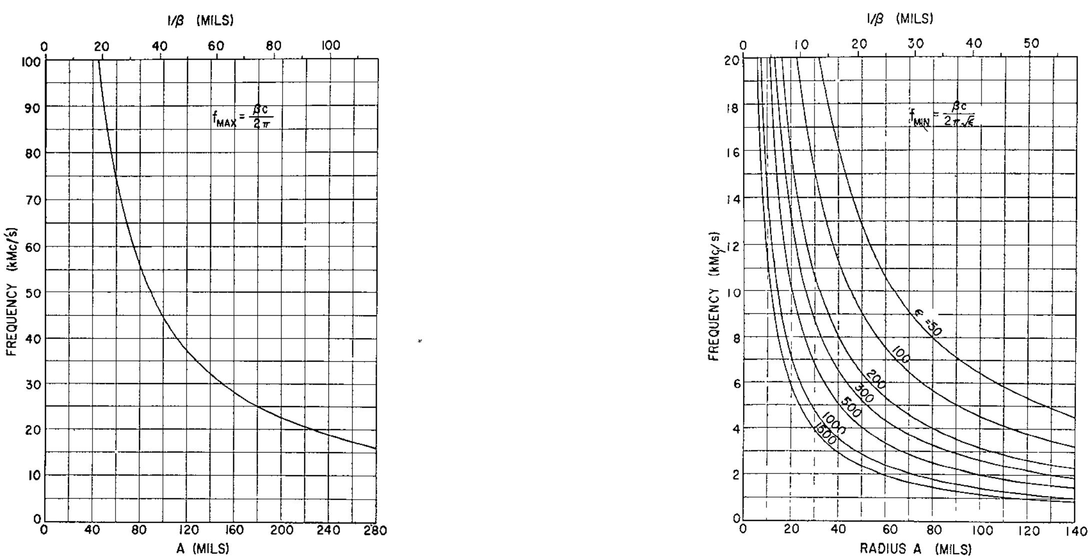
$$\tan^2\left(\frac{\beta L}{2}\sqrt{K_\epsilon^2 - 1}\right) = \frac{1 - K^2}{K_\epsilon^2 - 1} \tag{11}$$

where $K \equiv k/\beta$, $K_\epsilon \equiv k_\epsilon/\beta$ and k, k_ϵ are wave numbers in free space and in the dielectric medium, respectively. Limiting frequencies correspond to the poles and zeroes of (11). In principle, this expression may be used to compute resonant frequencies from known dielectric



(a) (b)

Fig. 1. (a) Rectangular parallelopiped dielectric resonator in the $TE_{11\delta}$ mode. (b) Circular cylindrical dielectric resonator in the $TE_{10\delta}$ mode.



(a) (b)

Fig. 2. (a) Maximum frequency for $TE_{11\delta}$ and $TE_{10\delta}$ modes. (b) Minimum frequency for $TE_{11\delta}$ and $TE_{10\delta}$ modes.

constants and physical dimensions. If the dielectric constant and physical dimensions are known, then calculation of the resonant frequency will in general require iteration of frequency. If ϵ, β, and f are assumed, then the length L may be calculated directly. The computer was programmed for the latter case. Equations (7) and (11) are valid for rectangular $TE_{11\delta}$ and cylindrical $TE_{10\delta}$ modes. (The extension to include higher order modes follows directly by redefining β and δ in these equations.) The asymptotic behavior of the resonant frequency also follows from (11). We find, that for low frequencies

$$f_{\text{low}} \propto \frac{1}{\sqrt{\epsilon}}$$

and for high frequencies

$$f_{\text{high}} \propto \frac{1}{\sqrt{\epsilon}\,L} \quad (\text{if } \beta c \ll \omega\sqrt{\epsilon}).$$

The lower frequencies are independent of L, the higher frequencies are inversely related to L. The asymptotic behavior of f with L and $\sqrt{\epsilon}$ is reflected in the curves of Figs. 3–8.

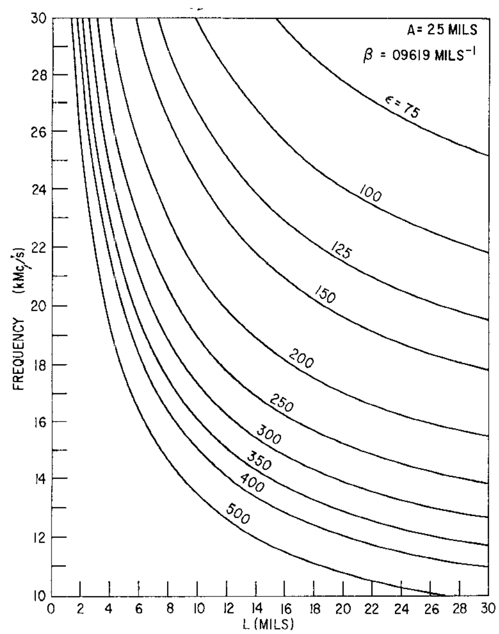


Fig. 3.

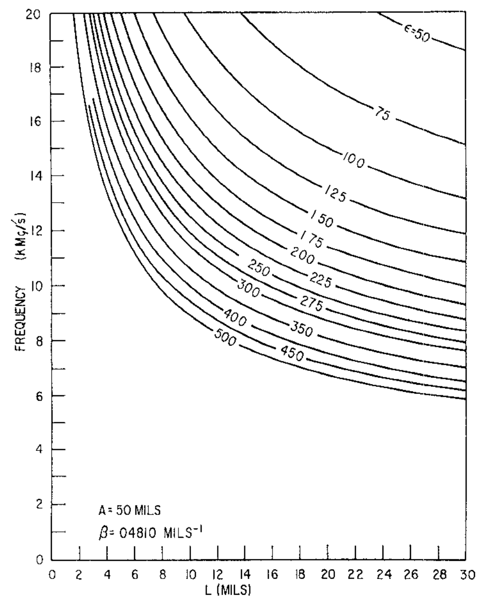


Fig. 4.

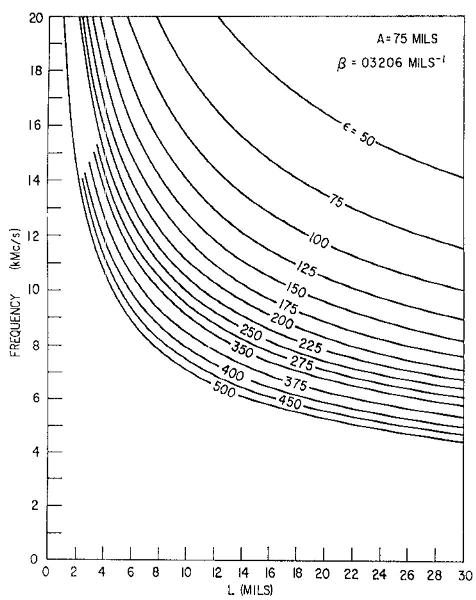


Fig. 5.

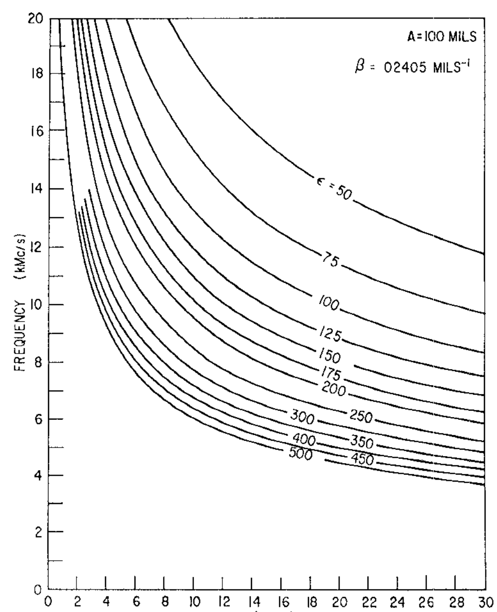


Fig. 6.

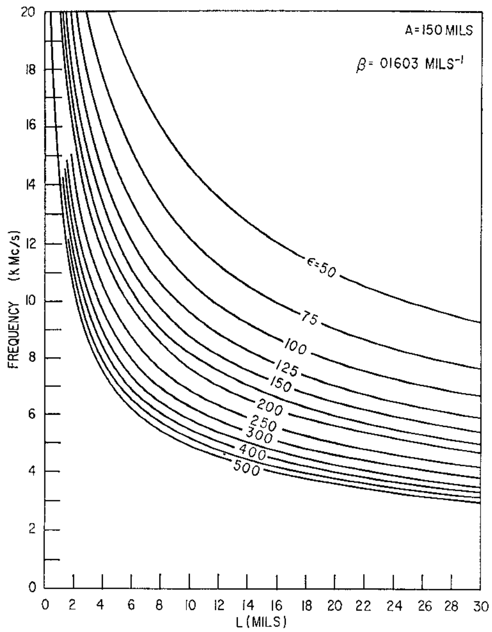


Fig. 7.

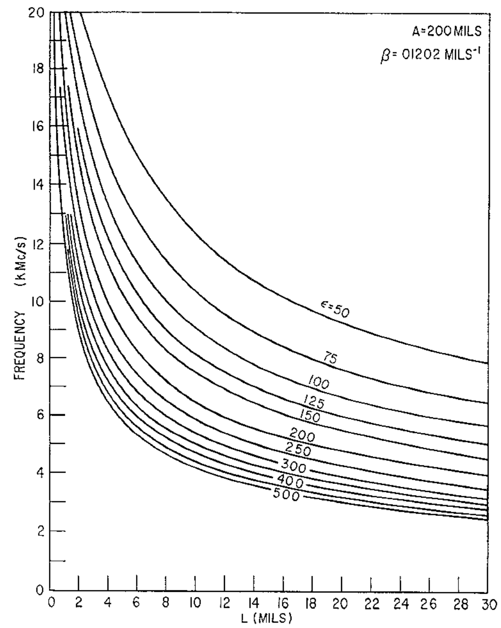


Fig. 8.

Figs. 3–8. Resonant frequency f vs. length L with ϵ as parameter for both cylindrical and rectangular resonators. The corresponding radius A for the cylindrical resonator is given on each figure. For the rectangular resonator, the corresponding dimensions A^* and B^* must be obtained from (6) or from Fig. 9 by using the value of β listed on each figure. There are an infinite number of permissible combinations of A^* and B^* for a given β.

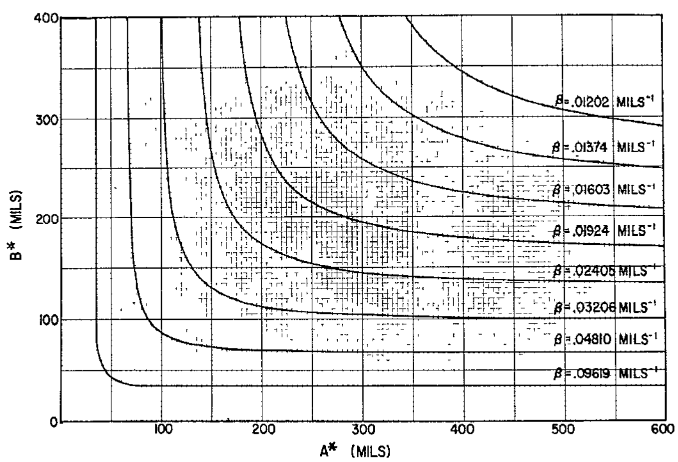


Fig. 9. Rectangular cross-sectional dimensions for parametric values of β.

Computer Program

The computer was programmed to calculate L for parametric values of cross-sectional dimensions, β, frequency, and relative dielectric constant. This procedure avoids the use of iteration, used by Yee, for calculating resonant frequencies. It did not, however, simplify or reduce the number of required calculations. In setting up the program it was found convenient to use Yee's quantities ζ and ζ_0, which follows.

$$L = \frac{2}{\zeta} \tan^{-1}\left(\frac{\zeta_0}{\zeta}\right) \tag{12}$$

where

$$\zeta^2 \equiv \left(\frac{\omega}{c}\right)^2 \epsilon - \beta^2 \tag{13}$$

$$\zeta_0^2 \equiv \beta^2 - \left(\frac{\omega}{c}\right)^2. \tag{14}$$

Values of $\beta, f,$ and ϵ were arbitrarily chosen; from them ζ and ζ_0 were calculated using (13) and (14). ζ and ζ_0 were then used to calculate L from (12). Final results are presented in Figs. 3–8. The resonant frequency is plotted vs. length on each of the graphs. Each graph contains a family of curves with each member corresponding to a particular value of ϵ. For each graph the resonator cross-sectional dimensions are given. Radius A of the cylindrical resonator is given explicitly; widths A^* and B^* of the rectangular resonator must be calculated from the given value of β [using (6)]. Alternatively A^* and B^* may be obtained graphically with the aid of Fig. 9. This is a graph of A^* vs. B^* for parametric values of β. The results shown in Figs. 3–8 cover a range of frequencies from zero to 30 kMc/s lengths from zero to 30 mils (1 mil $= 10^{-3}$ inch), ϵ from 50 to 500, A from 25 to 200 mils, and β from 0.01202 to 0.09619 mils^{-1}. This range of β includes cross-sectional widths from 40 to 600 mils. The actual program included a much larger range of values. The complete results will be presented in graphical and tabular form in an AFCRL Research Report [9]. The values given here hopefully include those of most practical interest in the microwave region at the present time.

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Impedances of Offset Parallel-Coupled Strip Transmission Lines

J. PAUL SHELTON, JR., MEMBER, IEEE

Abstract—**An offset parallel-coupled strip configuration is described, in which the mechanical parameters are strip width, strip offset, and ratio of strip spacing to ground-plane spacing. The electrical parameters are dielectric constant, characteristic impedance, even and odd mode impedance. The configuration is analyzed by conformal mapping techniques. Explicit design equations are derived in which the mechanical parameters are expressed in terms of the electrical parameters. Illustrative results are presented, and the limitations on coupling strength, characteristic impedance, and strip configuration are discussed.**

Introduction

Many MICROWAVE components are based upon parallel-coupled transmission-line sections. Examples are found among directional couplers, baluns, hybrid junctions, phase shifters, and filters [1]–[3]. In some cases, several coupled regions are used to obtain increased control over the theoretical characteristics of the component. In general, multisection components require different coupling values for the various sections. The subject of this paper is the analysis of a parallel-coupled strip transmission-line configuration that permits smooth variation of coupling from some designed maximum level to any lower value.

The first strip transmission-line technique for realizing variable coupling was edge-coupled coplanar strips [4]. The drawback to this method is the limitation of ρ, the ratio of even to odd mode impedance, to a maximum of two or three.

Getsinger proposed a technique for achieving very tight coupling in which a line with single center strip is sandwiched between the two strips of a second line [5]. This method requires the use of four layers of dielectric, and the transmission lines are unlike, one having single strip and one having double strips. Furthermore, it is somewhat difficult to determine the maximum coupling that is available for a given strip spacing.

Impedance relations for parallel strips, one above the other between ground planes, were derived by Cohn [6]. This configuration requires three dielectric layers and provides maximum coupling for given layer thicknesses. For this configuration, variation in coupling can be easily achieved in practice by offsetting the strips without changing the thicknesses of the dielectric layers. In general, both strip width and overlap are functions of

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The author is with the Institute for Defense Analyses, Arlington, Va. He was formerly with Radiation Systems, Inc., Alexandria, Va.